1st International Conference on Engineering and Technology Development (ICETD 2012)
Universitas Bandar Lampung
Faculty od Engineering and Faculty of Computer Science
ISSN 2301-6590

Paraboloid-Ellipsoid Programming Problem

Ismail Bin Mohd

Jabatan Matematik, Fakulti Sains dan Teknologi. Universiti Malaysia Terengganu

Mengabang Telipot, 21030 Kuala Terengganu, Malaysia

Abstract— **In this paper, we discuss the state-of-the-art models in estimating, evaluating, and selecting among non-linear mathematical models for obtaining the optimal solution of the optimization problems which involve the nonlinear functions in their constraints. We review theoretical and empirical issues including Newton's method, linear programming, quadratic programming, quadratically constrained programming, parabola, ellipse and the relation between parabola and ellipse. Finally, we outline our method called paraboloid-ellipsoid programming which is useful for solving economic forecasting and financial time-series with non-linear models.**

Keywords : Parabola, Ellipse, Optimization, Algorithm

I. Introduction

In optimization, we select the best alternative(s) from a set of alternatives by using **optimization approach** according to well defined objective criteria. Mathematical techniques are used to search the variables that give the maximum or minimum of the objective function.

Optimization techniques are extremely important for management and design. The techniques by themselves do not guarantee that the optimal alternative will be selected. To ensure selection of the optimal alternative, it must be included in the set of available choice of methods.

The objective function explains the essential characteristics of what is to be optimized. The function combines the essential descriptive quantitative variables. The limits of the values of variable for each alternative can be expressed as constraints on the range of values that may be used by an optimal alternative. The maximum or minimum criteria are chosen by the nature of the variables and objectives, and for examples, costs are minimized, and profits are maximized.

The most frequently used methods for searching the optimum value of a mathematical function are

a. differential calculus
b. search methods
c. direct method
d. mathematical (linear and nonlinear) programming
e. classical matrix method
f. calculus of variation
g. Bellman's Dynamic programming
h. Pontryagin's maximum principle

In this paper, a new method so-called paraboloid-ellipsoid programming for solving the special type nonlinear programming problem will be proposed. However this new method can be extended or modified for solving the other types of problems. In order to understand how this new method is proposed, we orderly arranged all the materials in several sections as follows. In Section 2, we briefly provide an explanation about Newton's method to be used in this paper. Linear programming and quadratic programming will be described in Section 3 and Section 4 respectively. Section 5 contains one of the quadratic programming problem where its constraints consist of quadratic function and a set of linear system. In Section 6, we need to expose to the reader about the parabola in great detail, and this is very useful in solving the problem which involves the conics. We continue the explanation about the ellipse in Section 7. A new standard ellipse which plays an important rule in this paper, is described in Section 8. Our new method will be explained in Section 9 and some numerical results will be displayed in Section 10 where its computation is done by using the algorithm given in Section 11. Conclusion given in Section 12 will end our paper.

II. Newton's Method

Newton's method (or **Newton–Raphson method**) ([1]) defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(n=0,1,2,…), (2.1)

is perhaps the best known method for searching successively better approximations to the zeros (or **roots**) of a **real**-valued **function**. Newton's method can often converge remarkably quickly, especially if the iteration (2.1) begins with x_0 by "sufficiently closed" to the desired root.

1. Linear Programming

A linear programming (LP) problem ([2]) is one in which the objective and all of the constraints are linear functions of the decision variables.

Since all linear functions are **convex**, linear programming problems are intrinsically easier to solve than general nonlinear (NLP) problems, which may be non-convex. In a **non-convex**, NLP there may be more than one feasible region and the optimal solution might be found at any point within any such region. In contrast, an LP has at most one feasible region with 'flat faces' (i.e. no curves) on its outer surface, and the optimal solution will always be found at a vertex (corner point) on the surface where the constraints intersect.

III. Quadratic Programming

A quadratic programming (QP) ([3]) which optimizes the quadratic objective subject to linear constraints, is widely used by the Markowitz mean-variance portfolio optimization problem, where the quadratic objective is the portfolio variance (sum of the variances and covariances of individual securities), and the linear constraints specify a lower bound for portfolio return.

If $x \in R^n$, the n×n matrix Q is symmetric, and c is any n×1 vector then QP is the problem which minimize

$$f(x) = \frac{1}{2} x^T Q x + c^T x$$

subject to

$$Ax \le b \text{ and } Ex = d$$

where “T” indicates the vector transpose.

QP problems, like LP problems, have only one feasible region with "flat faces" on its surface (due to the linear constraints), but the optimal solution may be found anywhere within the region or on its surface. The quadratic objective function may be **convex** which makes the problem easy to solve or **non-convex**, which makes it very difficult to solve.

If Q is a **positive semidefinite matrix,** then f(x) is a **convex function** ([4][5]). In this case the quadratic program has a global minimizer if there exists at least one vector x satisfying the constraints and f(x) is bounded below on the feasible region. If the matrix Q is **positive definite matrix,** then this global minimizer is unique. Portfolio optimization problems are usually of this type. If Q is zero, then the problem becomes a **linear program**. From optimization theory, a necessary condition for a point x to be a global minimizer is for it to satisfy the **Karush-Kuhn-Tucker** (KKT) conditions. The KKT conditions are also sufficient when f(x) is convex.

If there are only equality constraints, then the QP can be solved by a **linear system**. Otherwise, a variety of methods for solving the QP are commonly used, including **interior point**, **active set**, **exploration**, and **conjugate gradient** methods.

Convex quadratic programming is a special case of the more general field of **convex optimization**.

Complexity

For **positive definite** Q, the **ellipsoid method** solves the problem in **polynomial time**. If, on the other hand, Q is **negative definite**, then the problem is **NP-hard** ([5][6]). In fact, even if Q has only one negative **eigenvalue**, the problem is **NP-hard** ([6][7]). If the objective function is purely quadratic, negative semidefinite and has fixed rank, then the problem can be solved in **polynomial time** ([8]).

IV. Quadratically Constrained Quadratic Programming

In mathematics, a quadratically constrained quadratic programming (QCQP) is the problem of **optimizing** a quadratic objective function of the decision variables, and subject to constraints which are quadratic and linear functions of the variables ([9]). The problem is to minimize

$$\frac{1}{2} x^T P_0 x + q_0^T x$$

subject to

$$x^T P_i x + q_i^T x + r_i \le 0 \quad \text{for } i = 1, \dots, m,$$
$$Ax = b,$$

where P_0, … P_n are nxn matrices and $x \in R^n$ is the optimization variable. If P_1, … P_n are all zero, then the constraints are in fact linear and the problem is a **quadratic programming**.

Hardness

Solving the general case is an **NP-hard** problem. To see this, note that the two constraints $x_1(x_1 - 1) \le 0$ and $x_1(x_1 - 1) \ge 0$ are equivalent to the constraint $x_1(x_1 - 1) = 0$, which is in turn equivalent to the constraint $x_1 \in \{0,1\}$. Hence, any **0-1 integer programming** (in which all variables have to be either 0 or 1) can be formulated as a quadratically constrained quadratic programming. But 0–1 integer programming is NP-hard, so QCQP is also NP-hard.

V. Parabola

Fig. 6.1 shows the parabola ([10][11]) having the equation $y = kx^2$ where $0 < k < +\infty$. The focus and the directrix have the coordinates $(0, 1/4k)$ and the equation $y = -1/4k$ respectively.

It can be shown that the line through $P(x_1, kx_1^2)$ parallel to the axis of the parabola intersects the directrix at the point $D(x_1, -1/4k)$. We also can show that the tangent at $P(x_1, kx_1^2)$ intersects the axis of the parabola at the point $Q(0, -kx_1^2)$, and finally we can prove that the quadrilateral $QDPF$ is a (focal) rhombus.

Furthermore, the diagonals of this rhombus are perpendicular to each other and that they intersect at the point $(x_1/2, 0)$.

 ISSN 2301-6590

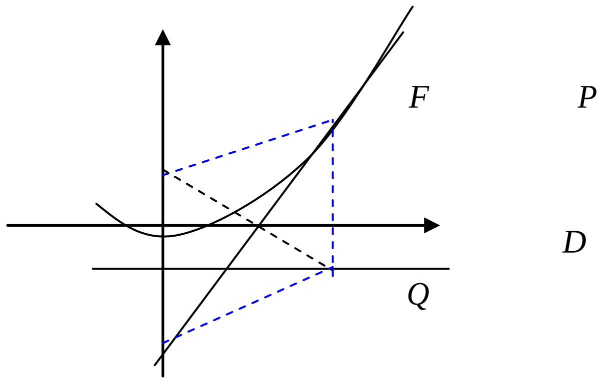

Fig. 6.1 : Parabola $y = kx^2$

VI. ELLIPSE

Fig. 7.1 shows us an ellipse ([12]) with some numerical dimensions where F_1 and F_2 are foci, L_1 and L_2 are directrices, a and b are major and minor axes of the ellips respectivey, and V_1 and V_2 are vertices of the ellipse. Both a and b are related in the form $b^2 = a^2(1-e^2)$.

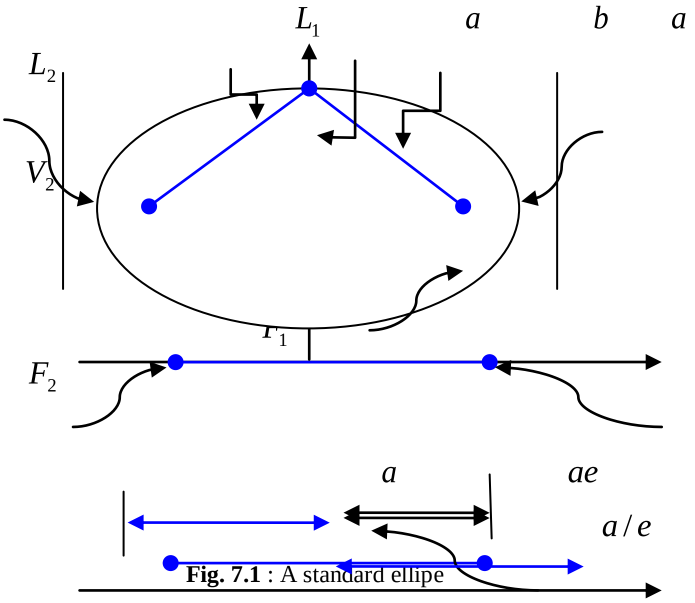

Fig. 7.1 : A standard ellipse

Th ellipse in Fig. 7.1 has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (7.1)$$

In this paper, we are dealing with more general ellipse defined by

$$\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1 \qquad (7.2)$$

where (p, q) is its centre.

Ellipse Family

For our purpose, we ned to consider two types of ellipse family where the **first** family deals with $b < a$ and the **second** family deals with $a < b$. However, the most important thing to be coniderd in tis paper, is the ellipses which have a common tangent line to the ellips at the point $(-x_1, kx_1^2)$ where $(0 < k < +\infty)$ and its gradient is $2kx_1$.

The Common Tangent Line

Suppose that we have given an ellipse family shown in Fig. 7.2 where $a < b$. Clearly that the equation of ellipse I can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The gradient of the ellipse at point $(-x_1, kx_1^2)$ is given by

$$\frac{dy}{dx} = \frac{x_1}{kx_1^2}\frac{b^2}{a^2}.$$

Since we want this gradient is equal to $2kx_1$ then we obtain

$$b^2 = 2k^2x_1^2a^2.$$

By substituting these b^2 and $(-x_1, kx_1^2)$ into the equation of ellipse I, we then obtain the equation

$$\frac{(-x_1)^2}{a^2} + \frac{(kx_1^2)^2}{2k^2x_1^2a^2} = 1$$

which can be solved to give

$$a^2 = \frac{3}{2}x_1^2 \text{ and } b^2 = 3k^2x_1^4.$$

Now our ellipse will have the following form

$$\frac{x^2}{\frac{3}{2}x_1^2} + \frac{y^2}{3k^2x_1^4} = 1. \qquad (7.3)$$

ISSN 2301-6590

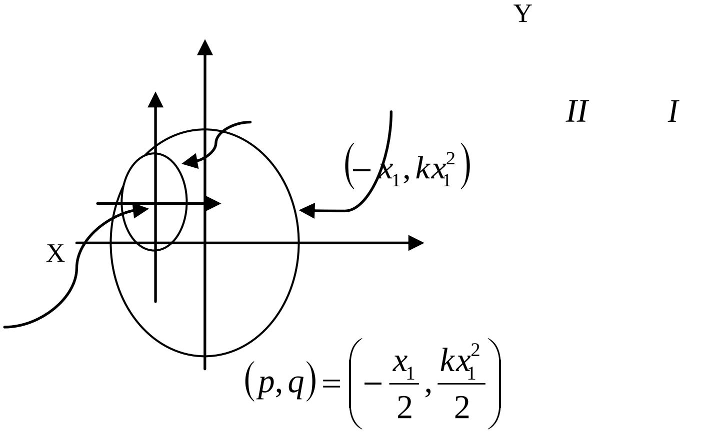

Fig. 7.2 : An ellipse family

By similar way, the equation of ellipse II can be written by

$$\frac{\left(x+\frac{x_1}{2}\right)^2}{\frac{3}{8}x_1^2}+\frac{\left(y-\frac{kx_1^2}{2}\right)^2}{\frac{3}{4}k^2x_1^4}=1$$

or

$$\frac{\left(x+\frac{x_1}{2}\right)^2}{\frac{3}{2}x_1^2}+\frac{\left(y-\frac{kx_1^2}{2}\right)^2}{3k^2x_1^4}=\frac{1}{4} \tag{7.4}$$

If we set $k=1,\ x_1=2$, then the equations (7.3) and (7.4) can be written as

$$\frac{x^2}{6}+\frac{y^2}{48}=1$$

(7.5) and

$$\frac{(x+1)^2}{6}+\frac{(y-2)^2}{48}=\frac{1}{4}.$$

(7.6) respectively.

What we have observed that the points

$$(0,0),\ \left(-\frac{x_1}{2}, \frac{kx_1^2}{2}\right),\ \left(-x_1, kx_1^2\right)$$

are collinear.

VII. A New Standard Ellipse

In this section, we will introduce a new standard ellipse related to parabola $y=kx^2$ where $0<k<+\infty$ and its configuration is shown in Fig. 8.1.

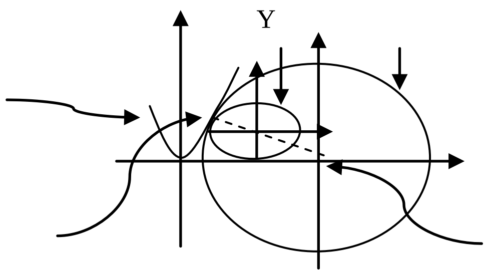

Fig. 8.1 : A new standard ellipse

Suppose that the parabola $y=kx^2$ touches the ellipse I which has the equation

$$\frac{(x-22)^2}{420}+\frac{y^2}{336}=1$$

We would like to obtain an ellipse II which touches the x-axis and parabola at $(p,0)$ and (2.4) repectively where its centre (p,b) is on the dash line

$$5y+x=22.$$

Clearly, the centre of this new ellipse can be computed as $(22-5b,b)$. Furthermore, by some manipulation, for this ellipse, we obtain the equation

$$\frac{(x-22+5b)^2}{a^2}+\frac{(y-b)^2}{b^2}=1$$

and by through some calculation we obtain

$$a=\frac{21-\sqrt{21}}{\sqrt{20}} \text{ and } b=\frac{21-\sqrt{21}}{5},$$

Finally, we have

$$\frac{\left(x-\left(1+\sqrt{21}\right)\right)^2}{\frac{5}{4}\left(\frac{21-\sqrt{21}}{5}\right)^2}+\frac{\left(y-\frac{21-\sqrt{21}}{5}\right)^2}{\left(\frac{21-\sqrt{21}}{5}\right)^2}=1$$

as the equation of the ellipse II so-called **new ellipse standard**.

VIII. Problem Statement

In this paper, we would like to develop two problems called paraboloid-ellipsoid programming problems which involve ellipse as an objective and parabola as its constraint. The first problem is defined as follows.

Minimize

$$z = \frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2}$$
(9.1)

subject to

$$y - kx^2 \geq 0, \text{ and } x, y \geq 0$$
(9.2)

where its configuration is given by Fig. 9.1 and its feasible region is inside the prabola of the first quadrant.

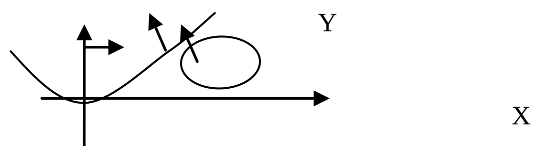

Fig. 9.1 : The first problem

The second problem is defined as follows.

Minimize

$$z = \frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2}$$
(9.3)

subject to

$$y - kx^2 \leq 0, \quad \text{and} \quad x, y \geq 0$$ (9.4)

where its configuration is given by Fig. 9.2 and its feasible region is outside the parabola of the first quadrant.

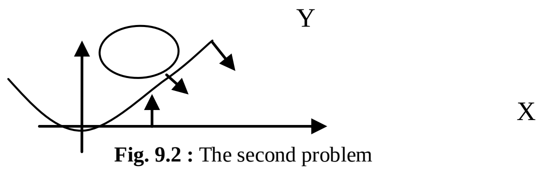

Fig. 9.2 : The second problem

Now, we are showing how to obtain the minimizer of the paraboloid-ellipsoid programming problem. Off course, we know the ellipse and te parabola. Therefore by using their equations, we have the gradient of ellipse at point (x, y) as

$$\frac{dy}{dx} = -\beta \frac{(x-p)}{(y-q)}$$
(9.5)

where $\beta = b^2 / a^2$. Since any point on the parabola is of the form (x, kx^2) and its gradient is given by $2kx$, then for (x_1, kx_1^2) we have

$$-\beta \frac{(x_1 - p)}{(kx_1^2 - q)} = 2kx_1$$
(9.6)

which can be simplified as

$$2k^2 x_1^3 - x_1(2kq - \beta) - p\beta = 0$$
(9.7)

from which the value of x_1 can be obtained.

If we substitute β by $1/\beta$ with $\beta = a^2 / b^2$, then we obtain

$$2\beta k^2 x_1^3 + x_1(1 - 2\beta kq) - p = 0$$
(9.8)

Example 9.1

Suppose that $a^2 = 9$, $b^2 = 4$, $k = 1$, $p = 4$ and $q = 3$. By using the above last formula we have

$$9x_1^3 - 25x_1 - 8 = 0$$

and when we solve to give $x_1 \approx 1.809$. Accordingly we will get the minimizer and the vlue of the objective.♦

2. Algorithm of the PEP Problem

Suppose that we would like to minimize

$$z = \frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2}$$

subject to standard form

↑ $y - kx^2 \geq 0$ $(k > 0)$ and $x, y \geq 0$

The algorithm to be used for solving the PEP problem is as follows.

Algorithm PEP

Data : p, q, k, max $\in$ R, $\beta = a^2 / b^2$, and $f(x_1) = 2\beta k^2 x_1^3 + x_1(1 - 2\beta kq) - p$

1. i = 0
2. **while** i < max **do**
   2.1. $fx_i = f(x_i)$
   2.2. $fdx_i = f'(x_i)$
   2.3. $x_{i+1} = x_i - \frac{fx_i}{fdx_i}$
   2.4. **if** x_{i+1} follows the Newton stopping criterium **then**
   2.4.1. **stop**

 ISSN 2301-6590

else

2.4.2. i = i+1

3. **return.** ♦

3. Numerical Result

Our Algorithm PEP has been tested by using the following examples.

Example 11.1

For this example we have used $a^2=9$, $b^2=4$, $p=4$, $q=3$, and , using $2\beta k^2x_1^3+x_1(1-2\beta kq)-p=0$ for $k=1,\ 2,\ 3,\ 0.5,\ 0.25,\ 0.1$.

k	1	2	3	0.5	0.25	0.1
x_1	1.809	1.272433	1.039729	2.		
x_1^2	3.272481	1.620358422	1.070754955	6.		
y_1	3.272481	3.240716845	3.212264864	3		
z	z_1	z_2	z_3			

z = (0.5519456686, 0.8408060509, 0.9882165103,0.2

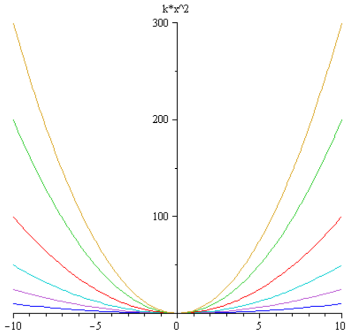

Fig. 11.2 : Variety of parabolas with different k

The relationship between objective functions and their constraints for Example 11.1 are drawn in Fig. 11.3.

The Fig.s of all ellipses listed in Example 11.1 are drawn in Fig. 11.1 where their radii are given by vector Z .

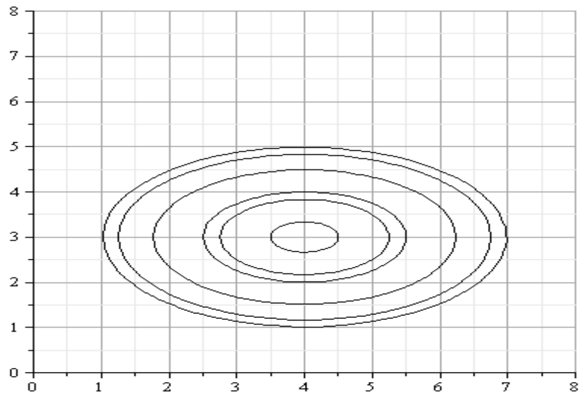

Fig. 11.1 : Variety of ellipse with different radius

The Fig.s of all parabolas listed in Example 11.1 are drawn in Fig. 11.2 where their k's are given by k's row.

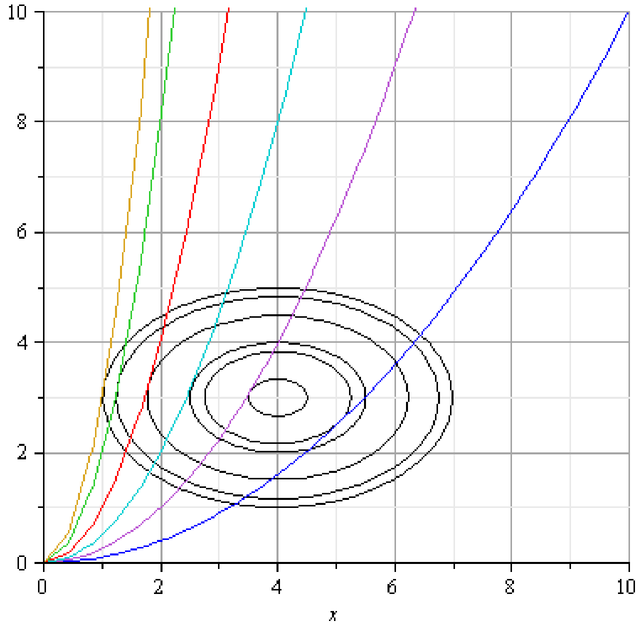

Fig. 11.3 : Relationship beteen parabolas and ellipses for Example 11.1.

IX. Discussion

The Algorithm PEP which based on Newton's method and the formula given by (9.8) can be used to obtain x_1, the solution of paraboloid-ellipsoid programming which happened at point (x_1, kx_1^2).

By using (9.8) with $\beta=a^2/b^2$, we can obtain the centre point of new objective ellipse on x-axis given by

$$p=x_1+2\beta k^2x_1^2=x_1(1+2\beta k^2x_1)$$

$(\beta=a^2/b^2)$ (12.1)

From (12.1) we have the following piece of programming.

1. **if** $\beta>1$

then

1.1. we have an ellipse with major and minor axis given by a and b respectively

else

1.2. we have an ellipse with major and minor axis given by b and a respectively

2. **return**. ♦

X. Conclusion

We have shown that both parabola and ellipse have some relationship feastures which can be exploited for obtaining the solution(s) of the economic problems of the paraboloid-ellipsoid programming.

Although we can find this relationship percisely, we still use the approximated method (in this paper Newton's method) to obtain the solution, and therefore in order to obtain more precise result we need to seek the best criterium for stopping the routine in Newton's method.

Our method can be extended to the problem with more than one constraint and we prefer to explain in another paper.

ACKNOWLEDGEMENT

The author would like to express his gratitude to Ridwan Pandiya and Herlina Napitupulu for their assistance in the preparation of the manuscript. The author also extends his appreciation to the Universiti Malaysia Terengganu for their support in this research

REFERENCES

[1] http://en.wikipedia.org/Newton's method
[2] http://en.wikipedia.org/Linear programming
[3] http://en.wikipedia.org/Quadratic programming
[4] Kozlov, M.K.; Tarasov, S.P.; Khachiyan, L.G. "Polynomial solvability of convex quadratic programming," in Sov. Math., Dokl. 20, 1108-1111 (1979). This is a translation from Dokl. Akad. Nauk SSSR 248, 1049-1051 (1979). ISSN: 0197-6788.
[5] Sahni, S. "Computationally related problems," in SIAM Journal on Computing, 3, 262--279, 1974.
[6] Michael R. Garey and David S. Johnson (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman. ISBN 0-7167-1045-5. A6: MP2, pg.245.
[7] Panos M. Pardalos and Stephen A., Quadratic programming with one negative eigenvalue is NP-hard, Vavasis in Journal of Global Optimization, Volume 1, Number 1, 1991, pg.15-22.
[8] Allemand, K. (Doctoral thesis 2496, 'Optimisation quadratique en variables binaires : heuristiques et cas polynomiaux'), Swiss Federal Institute of Technology, www.epfl.ch
[9] Stephen Boyd and Lieven Vandenberghe, Convex Optimization (book in pdf)
[10] Lockwood, E. H. "The Parabola." Ch. 1 in *A Book of Curves.* Cambridge, England: Cambridge University Press, pp. 2-12, 1967.
[11] Loomis, E. S. "The Parabola." §2.5 in *The Pythagorean Proposition: Its Demonstrations Analyzed and Classified and Bibliography of Sources for Data of the Four Kinds of "Proofs," 2nd ed.* Reston, VA: National Council of Teachers of Mathematics, pp. 25-28, 1968.
[12] Ralph, P. A., "Calculus : Analytic Geometry and Calculus, with Vector", New York, USA, McGraw-Hill, 1962.